

Refinement of Gini-Means Inequalities and Connections with Divergence Measures

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Abstract

In 1938, Gini [3] studied a mean having two parameters. Later, many authors studied properties of this mean. It contains as particular cases the famous means such as *harmonic*, *geometric*, *arithmetic*, etc. Also it contains, the *power mean of order r* and *Lehmer mean* as particular cases. In this paper we have considered inequalities arising due to Gini-Mean and Heron's mean, and improved them based on the results recently studied by the author [13].

Key words: *Arithmetic mean; Geometric Mean; Harmonic Mean; Gini Mean; Power Mean; Refinement inequalities*

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1 Gini Mean of order r and s

The Gini [3] mean of order r and s is given by

$$E_{r,s}(a, b) = \begin{cases} \left(\frac{a^r + b^r}{a^s + b^s} \right)^{\frac{1}{r-s}} & r \neq s \\ \exp \left(\frac{a^r \ln a + b^r \ln b}{a^r + b^r} \right) & r = s \neq 0 \\ \sqrt{ab} & r = s = 0 \end{cases} \quad (1)$$

In particular when $s = 0$ in (1), we have

$$E_{r,0}(a, b) := B_r(a, b) = \begin{cases} \left(\frac{a^r + b^r}{2} \right)^{\frac{1}{r}}, & r \neq 0 \\ \sqrt{ab}, & r = 0 \end{cases} \quad (2)$$

Again, when $s = r - 1$ in (1), we have

$$E_{r,r-1}(a, b) := K_s(a, b) = \frac{a^r + b^r}{a^{r-1} + b^{r-1}}, \quad r \in \mathbb{R} \quad (3)$$

The expression (2) is famous as *mean of order r* or *power mean*. The expression (3) is known as *Lehmer mean*[4]. Both these means are monotonically increasing in r . Moreover, these two have the following inequality [2] among each other:

$$B_r(a, b) \begin{cases} < L_r(a, b), & r > 1 \\ > L_r(a, b), & r < 1 \end{cases} \quad (4)$$

Since $E_{r,s} = E_{s,r}$, the Gini-mean $E_{r,s}(a, b)$ given by (1) is an increasing function in r or s . Using the monotonicity property [1], [7], [5] we have the following inequalities:

$$\begin{aligned} E_{-3,-2} &\leq E_{-2,-1} \leq E_{-3/2,-1/2} \leq E_{-1,0} \leq E_{-1/2,0} \leq \\ E_{-1/2,1/2} &\leq E_{0,1/2} \leq E_{0,1} \leq \{E_{0,2} \text{ or } E_{1/2,1}\} \leq E_{1,2}. \end{aligned} \quad (5)$$

Let us write the expression (5) as

$$P_1 \leq P_2 \leq P_3 \leq H \leq P_4 \leq G \leq N_1 \leq A \leq (P_5 \text{ or } S) \leq P_6, \quad (6)$$

where $P_1 = E_{-3,-2} = K_{-2}$, $P_2 = E_{-2,-1} = K_{-1}$, $P_3 = E_{-3/2,-1/2} = K_{-1/2}$, $H = E_{-1,0} = K_0 = B_{-1}$, $P_4 = E_{-1/2,0} = B_{-1/2}$, $G = E_{-1/2,1/2} = K_{1/2} = B_0$, $N_1 = E_{0,1/2} = B_{1/2}$, $A = E_{0,1} = K_1 = B_1$, $P_5 = E_{1/2,1}$, $B_2 = E_{0,2} = S$ and $P_6 = E_{1,2} = K_2$.

The means H , G , A and S are the *harmonic*, *geometric*, *arithmetic* and the *square-root* means respectively. In [10, 12], the author studied the following inequalities:

$$H \leq G \leq N_1 \leq N_3 \leq N_2 \leq A \leq S, \quad (7)$$

where

$$N_2(a, b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right) \left(\sqrt{\frac{a+b}{2}} \right)$$

and

$$N_3(a, b) = \frac{a + \sqrt{ab} + b}{3}.$$

The expression $N_3(a, b)$ is famous as Heron's mean. Some applications of the inequalities (7) can be seen in [8], [6]. Combining (6) and (7), we have the following sequence of inequalities:

$$P_1 \leq P_2 \leq P_3 \leq H \leq P_4 \leq G \leq N_1 \leq N_3 \leq N_2 \leq A \leq \{P_5 \text{ or } S\} \leq P_6. \quad (8)$$

The expression (8) admits many non-negative differences. Let us write them as follows:

$$D_{tp}(a, b) = bg_{tp}\left(\frac{a}{b}\right) = b \left[f_t\left(\frac{a}{b}\right) - f_p\left(\frac{a}{b}\right) \right], \quad (9)$$

where

$$g_{tp}(x) = f_t(x) - f_p(x), \quad f_t(x) \geq f_p(x), \quad \forall x > 0.$$

More precisely, the function $f : (0, \infty) \rightarrow \mathbb{R}$ appearing in (9) lead us to the following inequalities:

$$\begin{aligned} f_{P_1}(x) &\leq f_{P_2}(x) \leq f_{P_3}(x) \leq f_H(x) \leq f_{P_4}(x) \leq f_G(x) \leq f_{N_1}(x) \leq \\ &\leq f_{N_3}(x) \leq f_{N_2}(x) \leq f_A(x) \leq \{f_{P_5}(x) \text{ or } f_S(x)\} \leq f_{P_6}(x). \end{aligned}$$

Equivalently, we have

$$\begin{aligned} \frac{x(x^2+1)}{x^3+1} &\leq \frac{x(x+1)}{x^2+1} \leq \frac{x(\sqrt{x}+1)}{x^{3/2}+1} \leq \frac{2x}{1+x} \leq \frac{4x}{(\sqrt{x}+1)^2} \leq \\ &\leq \sqrt{x} \leq \left(\frac{\sqrt{x}+1}{2}\right)^2 \leq \frac{x+\sqrt{x}+1}{3} \leq \left(\frac{\sqrt{x}+1}{2}\right) \left(\sqrt{\frac{x+1}{2}}\right) \leq \\ &\leq \frac{x+1}{2} \leq \left\{ \left(\frac{x+1}{\sqrt{x}+1}\right)^2 \text{ or } \sqrt{\frac{x^2+1}{2}} \right\} \leq \frac{x^2+1}{x+1}. \end{aligned} \quad (10)$$

Based on the differences arising due to inequalities (8) written according to (9), with the property that the functions are convex, the author [13] proved the following sequences of inequalities:

$$\begin{aligned} \frac{1}{8}D_{P_6P_1} &\leq \frac{1}{6}D_{P_6P_2} \leq D_{SA} \leq \frac{1}{3}D_{SH} \leq \frac{1}{2}D_{AH} \leq \\ &\leq \left\{ \left\{ \frac{4}{9}D_{P_6N_2} \right\} \leq \left\{ \frac{3}{7}D_{P_6N_3} \right\} \leq \left\{ \frac{2}{5}D_{P_6N_1} \right\} \right\} \leq \frac{1}{3}D_{P_6G} \leq \left\{ \frac{2}{3}D_{P_5H} \right\} \leq 4D_{N_2N_1} \leq \\ &\leq \frac{4}{3}D_{N_2G} \leq D_{AG} \leq 4D_{AN_2} \leq \frac{2}{3}D_{P_5G} \leq D_{P_5N_1} \leq \\ &\leq \frac{6}{5}D_{P_5N_3} \leq \frac{4}{3}D_{P_5N_2} \leq 2D_{P_5A}, \end{aligned} \quad (11)$$

$$D_{SA} \leq \left\{ \frac{4}{3}D_{SN_2} \right\} \leq \frac{2}{3}D_{SN_1} \leq \left\{ \frac{1}{3}D_{P_6G} \right\} \leq \frac{2}{5}D_{P_5H}, \quad (12)$$

and

$$\left\{ \frac{1}{8}D_{P_6P_1} \right\} \leq \left\{ \frac{1}{6}D_{P_6P_2} \right\} \leq \frac{2}{7}D_{P_5P_3} \leq \frac{4}{9}D_{P_6N_2} \leq D_{P_6S} \leq D_{AG}. \quad (13)$$

where, for example, $D_{P_6N_2} = P_6 - N_2$, $D_{AG} = A - G$, $D_{P_5A} = P_5 - A$, etc.

Notation: Throughout the paper, the notation $A \leq \left\{ \begin{matrix} B \\ C \end{matrix} \right\}$ is understood as $A \leq B$ and $A \leq C$, but there is no relation between B and C .

The aim of this paper is to improve the inequalities (8) based on the results appearing in the inequalities (11)-(13).

2 Refinement Inequalities

The results appearing in the inequalities (11)-(13) lead us to the following two groups of individual inequalities:

Group 1:

1. $P_2 \leq \frac{P_6 + 3P_1}{4}.$	8. $G \leq \frac{P_6 + 6P_4}{7}.$	15. $\frac{S + 4N_2}{5} \leq A.$
2. $\frac{2S + H}{3} \leq A.$	9. $\frac{2N_2 + G}{3} \leq N_1.$	16. $\frac{S + 3N_3}{4} \leq A.$
3. $\frac{P_6 + 14N_1}{15} \leq N_3.$	10. $N_2 \leq \frac{3A + G}{4}.$	17. $\frac{S + 5N_1}{6} \leq N_2.$
4. $\frac{P_6 + 2P_4}{3} \leq N_3.$	11. $N_1 \leq \frac{P_5 + 2G}{3}.$	18. $\frac{S + 8N_1}{9} \leq N_3.$
5. $S \leq \frac{5P_6 + 2P_4}{7}.$	12. $N_3 \leq \frac{P_5 + 5N_1}{6}.$	19. $\frac{S + 3G}{4} \leq N_1.$
6. $\frac{P_6 + 3G}{4} \leq N_2.$	13. $N_2 \leq \frac{P_5 + 9N_3}{10}.$	20. $P_2 \leq \frac{9P_1 + 4P_5}{13}.$
7. $\frac{P_6 + 5G}{6} \leq N_1.$	14. $A \leq \frac{P_5 + 2N_2}{3}.$	21. $P_3 \leq \frac{2P_5 + 7P_2}{9}.$
		22. $S \leq \frac{5P_6 + 4N_2}{9}.$

Group 2:

1. $6A + P_6 \leq 6S + P_2.$	9. $A + 6N_1 \leq P_4 + 6N_2.$
2. $9A + 8N_2 \leq 9H + 8P_6.$	10. $G + 6A \leq P_5 + 6N_2.$
3. $7A + 6N_3 \leq 7H + 6P_6.$	11. $G + 2S \leq P_6 + 2N_1.$
4. $5A + 4P_4 \leq 5H + 4S.$	12. $4H + 5S \leq 4P_5 + 5G.$
6. $S + N_1 \leq P_4 + P_6.$	13. $16P_2 + 9P_6 \leq 16P_5 + 9P_1.$
6. $6H + 5P_6 \leq 6P_5 + 5G.$	14. $13P_2 + 12P_5 \leq 13P_6 + 12P_1.$
7. $2P_4 + P_6 \leq 2A + G.$	15. $12P_3 + 7P_6 \leq 12P_5 + 7P_2.$
8. $10N_1 + P_5 \leq 10N_2 + H.$	16. $14N_2 + 9P_5 \leq 14P_6 + 9P_3.$
	17. $P_6 + G \leq A + S.$

Based on the inequalities appearing in Group 1, we have the theorem giving the refinement of the inequalities appearing in (8).

Theorem 2.1. *The following inequalities hold:*

$$\begin{aligned}
G &\leq \frac{P_6+6P_4}{7} \leq \left\{ \frac{\frac{P_6+5G}{6}}{\frac{S+3G}{4}} \right\} \leq \frac{2N_2+G}{3} \leq N_1 \leq \left\{ \frac{\frac{10N_2+H-P_5}{10}}{\frac{P_4+6N_2-A}{6}} \right\} \leq \\
&\leq \left\{ \begin{array}{l} \left\{ \frac{\frac{P_5+2G}{3}}{P_4+P_6-S} \right\} \leq \frac{P_6+2P_4}{3} \\ P_4+P_6-S \left\{ \frac{\frac{P_6+14N_1}{15}}{\frac{S+8N_1}{9}} \right\} \end{array} \right\} \leq N_3 \leq \\
&\leq N_2 \leq \left\{ \begin{array}{l} \frac{3A+G}{4} \leq \frac{P_5+9N_3}{10} \leq \left\{ \frac{\frac{8P_6+9H-9A}{8}}{\frac{S+4N_2}{5}} \right\} \leq A \\ \frac{14P_6+9P_3-9P_5}{14} \end{array} \right\} \leq \\
&\leq \left\{ \frac{\frac{5H+4S-4P_4}{5}}{\frac{P_5+6N_2-G}{6}} \leq \frac{P_5+2N_2}{3} \leq \frac{7H+6P_6-6N_3}{7} \right\} \leq \\
&\leq \left\{ \begin{array}{l} \frac{4H+5S-5G}{4} \leq P_5 \\ S \leq \frac{P_6+2N_1-G}{2} \leq \left\{ \frac{\frac{2P_4+5P_6}{7}}{\frac{4N_2+5P_6}{9}} \right\} \leq P_6 \\ \frac{12P_5+13P_2-12P_1}{13} \end{array} \right\} \leq \\
&\leq \left\{ \frac{A+S-G}{\frac{6P_5+5G-6H}{5}} \right\} \leq \left\{ \frac{\frac{12P_5+7P_2-12P_3}{7}}{6S+P_2-6A} \right\} \leq \frac{16P_5+9P_1-16P_2}{9} \quad (14)
\end{aligned}$$

and

$$P_2 \leq \left\{ \begin{array}{l} \frac{P_6+3P_1}{4} \leq N_1 \leq \frac{P_6+3G}{4} \leq \frac{S+5N_1}{6} \leq N_2 \\ \frac{4P_5+9P_1}{13} \leq N_3 \leq \frac{P_5+5N_1}{6} \leq A \leq \frac{3I+2P_4}{2} \leq \left\{ \begin{array}{l} P_5 \leq \frac{T+2A}{2} \leq \frac{3J+16G}{16} \\ S \end{array} \right. \\ P_3 \leq \frac{2P_5+7P_2}{9} \leq N_1 \end{array} \right. \quad (15)$$

Proof. Some of the results appearing in (14) and (15) are either due to Groups 1 and 2 or are obvious. Here we shall prove only those are not obvious.

1. **For $\frac{P_6+6P_4}{7} \leq \frac{S+3G}{4}$:** We have to show that

$$\frac{1}{28} (7S + 21G - 4P_6 - 24P_4) \geq 0.$$

We can write $7S + 21G - 4P_6 - 24P_4 = \frac{1}{2}b g_1(a/b)$, where

$$g_1(x) = \frac{u_1(x)}{(x+1)(\sqrt{x}+1)^2},$$

with

$$\begin{aligned}
u_1(x) &= 7\sqrt{2x^2+2}(\sqrt{x}+1)^2(x+1) \\
&\quad - (8x^3 - 26x^{5/2} + 116x^2 - 84x^{3/2} + 116x - 26\sqrt{x} + 8).
\end{aligned}$$

Now we shall show that $u_1(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_1(x) &= \left[7\sqrt{2x^2 + 2} (\sqrt{x} + 1)^2 (x + 1) \right]^2 \\ &\quad - (8x^3 - 26x^{5/2} + 116x^2 - 84x^{3/2} + 116x - 26\sqrt{x} + 8)^2 \\ &= 2(\sqrt{x} - 1)^2 \left(\begin{aligned} &8\sqrt{x}(x^3 + 92x^2 + 92x + 1)(\sqrt{x} - 1)^2 + \\ &17x^5 + 430x^{9/2} + x^4 + 3064x^{7/2} + x + \\ &+ 5522x^{5/2} + 3064x^{3/2} + 430\sqrt{x} + 17 \end{aligned} \right). \end{aligned}$$

Since $v_1(x) \geq 0$ giving $u_1(x) \geq 0, \forall x > 0$, hence proving the required result.

Argument: Let a and b two positive numbers, i.e., $a > 0$ and $b > 0$. If $a^2 - b^2 \geq 0$, then we can conclude that $a \geq b$ because $a - b = (a^2 - b^2)/(a + b)$. If $b < 0$, then obviously, $a - (-b) = a + b > 0$ always holds. In order to apply the above argument, it is sufficient that $a > 0$. We have used this argument to prove $u_1(x) \geq 0, \forall x > 0$. We shall use frequently this argument to prove the other parts.

2. **For $\frac{S+3G}{4} \leq \frac{2N_2+G}{3}$:** We have to show that

$$\frac{1}{12} (8N_2 - 5G - 3S) \geq 0.$$

We can write $8N_2 - 5G - 3S = \frac{1}{2}b g_2(a/b)$, where

$$g_2(x) = 4(\sqrt{x} + 1) \sqrt{2x + 2} - (10\sqrt{x} + 3\sqrt{2x^2 + 2}).$$

Now we shall show that $g_2(x) \geq 0, \forall x > 0$. In order to prove it we shall apply twice the argument given in part 2 of section 3. Let us consider

$$\begin{aligned} v_2(x) &= \left[4(\sqrt{x} + 1) \sqrt{2x + 2} \right]^2 - (10\sqrt{x} + 3\sqrt{2x^2 + 2})^2 \\ &= 14x^2 + 46x^{3/2} + 18\sqrt{x}(\sqrt{x} - 1)^2 + 46\sqrt{x} + 14 \\ &\quad - 60\sqrt{x}\sqrt{2x^2 + 2}. \end{aligned}$$

Let us consider again

$$\begin{aligned} v_{2a}(x) &= \left[14x^2 + 46x^{3/2} + 18\sqrt{x}(\sqrt{x} - 1)^2 + 46\sqrt{x} + 14 \right]^2 \\ &\quad - \left[60\sqrt{x}\sqrt{2x^2 + 2} \right]^2 \\ &= 4(\sqrt{x} - 1)^4 (49x^2 + 644x^{3/2} + 1254x + 644\sqrt{x} + 49). \end{aligned}$$

Since $v_{2a}(x) \geq 0$, giving $v_2(x) \geq 0, \forall x > 0$. This implies that $g_2(x) \geq 0, \forall x > 0$, hence proving the required result.

3. **For $\frac{\mathbf{P}_6 + 5\mathbf{G}}{6} \leq \frac{2\mathbf{N}_2 + \mathbf{G}}{3}$:** We have to show that

$$\frac{1}{6} (4N_2 - 3G - P_6) \geq 0.$$

We can write $4N_2 - 3G - P_6 = b g_3(a/b)$, where

$$g_3(x) = \frac{u_3(x)}{x+1},$$

with

$$u_3(x) = \sqrt{2x+2} (\sqrt{x}+1) (x+1) - (3x^{3/2} + 3\sqrt{x} + x^2 + 1).$$

Now we shall show that $u_3(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_3(x) &= \left[\sqrt{2x+2} (\sqrt{x}+1) (x+1) \right]^2 - (3x^{3/2} + 3\sqrt{x} + x^2 + 1)^2 \\ &= (\sqrt{x}-1)^2 (x^2 + 2x^{3/2} + 2\sqrt{x} + x + 1). \end{aligned}$$

Since $v_3(x) \geq 0$, giving $u_3(x) \geq 0, \forall x > 0$, hence proving the required result.

4. **For $\frac{10\mathbf{N}_2 + \mathbf{H} - \mathbf{P}_5}{10} \leq \mathbf{P}_4 + \mathbf{P}_6 - \mathbf{S}$:** We have to show that

$$\frac{1}{10} (10P_4 + 10P_6 + P_5 - 10N_2 - 10S - H) \geq 0.$$

We can write $10P_4 + 10P_6 + P_5 - 10N_2 - 10S - H = \frac{1}{2} b g_4(a/b)$, where

$$g_4(x) = \frac{u_4(x)}{(\sqrt{x}+1)^2 (x+1)},$$

with

$$\begin{aligned} u_4(x) &= 22x^3 + 40x^{5/2} + 98x^2 + 4x (\sqrt{x}-1)^2 + 98x + 40\sqrt{x} + 22 \\ &\quad - 5 (\sqrt{x}+1)^2 (x+1) \left[2\sqrt{2x^2+2} + 5 (\sqrt{x}+1) \sqrt{2x+2} \right]. \end{aligned}$$

Now we shall show that $u_4(x) \geq 0, \forall x > 0$. In order to prove it we shall apply twice the argument given in Part 1. Let us consider

$$\begin{aligned} v_4(x) &= \left[22x^3 + 40x^{5/2} + 98x^2 + 4x (\sqrt{x}-1)^2 + 98x + 40\sqrt{x} + 22 \right]^2 \\ &\quad - \left\{ 5 (\sqrt{x}+1)^2 (x+1) \left[2\sqrt{2x^2+2} + 5 (\sqrt{x}+1) \sqrt{2x+2} \right] \right\}^2 \\ &= \left(\begin{aligned} &234 + 17240x^3 + 888x^{5/2} + 8102x^2 + 3508x^{3/2} + \\ &+ 3588x + 660\sqrt{x} + 234x^6 + 660x^{11/2} + \\ &+ 3588x^5 + 3508x^{9/2} + 8102x^4 + 888x^{7/2} \end{aligned} \right) \\ &\quad - 100\sqrt{2x^2+2}\sqrt{2x+2} \left(\begin{aligned} &26x^2 + 12x^{7/2} + 5x^4 + \\ &+ 26x^{5/2} + 12x + 20x^3 + \\ &+ 20x^{3/2} + x^{9/2} + 5\sqrt{x} + 1 \end{aligned} \right). \end{aligned}$$

Let us consider again

$$\begin{aligned}
v_{4a}(x) &= \left(\begin{aligned} &234 + 17240x^3 + 888x^{5/2} + 8102x^2 + 3508x^{3/2} + \\ &+ 3588x + 660\sqrt{x} + 234x^6 + 660x^{11/2} + \\ &+ 3588x^5 + 3508x^{9/2} + 8102x^4 + 888x^{7/2} \end{aligned} \right)^2 \\
&\quad - \left[100\sqrt{2x^2+2}\sqrt{2x+2} \left(\begin{aligned} &26x^2 + 12x^{7/2} + 5x^4 + \\ &+ 26x^{5/2} + 12x + 20x^3 + \\ &+ 20x^{3/2} + x^{9/2} + 5\sqrt{x} + 1 \end{aligned} \right) \right]^2 \\
&= 4(\sqrt{x}-1)^4 w_4(x),
\end{aligned}$$

where

$$w_4(x) = \left(\begin{aligned} &3689 - 144760x^{3/2} - 8024x^{19/2} - 8024\sqrt{x} + \\ &+ 402389x^2 + 6593432x^3 + 3689x^{10} - 25534x^9 + \\ &+ 402389x^8 + 6593432x^7 + 17426946x^6 + \\ &+ 26278348x^5 + 17426946x^4 + 1834912x^{15/2} + \\ &+ 10215648x^{13/2} + 18146128x^{11/2} + 18146128x^{9/2} + \\ &+ 10215648x^{7/2} - 25534x + 1834912x^{5/2} - 144760x^{17/2} \end{aligned} \right).$$

Now we shall show that $w_4(x) > 0$, $\forall x > 0$. Let us consider

$$h_4(t) = w_4(t^2) = \left(\begin{aligned} &3689t^{20} - 8024t^{19} - 25534t^{18} - 144760t^{17} + \\ &+ 402389t^{16} + 1834912t^{15} + 6593432t^{14} + \\ &+ 10215648t^{13} + 17426946t^{12} + 18146128t^{11} + \\ &+ 26278348t^{10} + 18146128t^9 + 17426946t^8 + \\ &+ 10215648t^7 + 6593432t^6 + 1834912t^5 + \\ &+ 402389t^4 - 144760t^3 - 25534t^2 - 8024t + 3689 \end{aligned} \right).$$

The polynomial equation $h_4(t) = 0$ of 20^{th} degree admits 20 solutions. All of them are convex (not written here). This means that there are no real positive solutions of the equation $h_4(t) = 0$. Thus we conclude that either $h_4(t) > 0$ or $h_4(t) < 0$, for all $t > 0$. In order to check it is sufficient to see for any particular value of $h_4(t)$, for example when $t = 1$. This gives $h_4(1) = 135168000$, hereby proving that $h_4(t) > 0$ for all $t > 0$, consequently, $v_{4a}(x) \geq 0$, for all $x > 0$ giving $v_4(x) \geq 0$, for all $x > 0$. Finally, proving the required result.

5. **For $\frac{P_4+6N_2-A}{6} \leq P_4 + P_6 - S$:** We have to show that

$$\frac{1}{6}(5P_4 + 6P_6 + A - 6S - 6N_2) \geq 0.$$

We can write $5P_4 + 6P_6 + A - 6S - 6N_2 = \frac{1}{2}b g_5(a/b)$, where

$$g_5(x) = \frac{u_5(x)}{(\sqrt{x}+1)^2(x+1)},$$

with

$$u_5(x) = 13x^3 + 26x^{5/2} + 55x^2 + 4x^{3/2} + 55x + 26\sqrt{x} + 13 \\ - 3(x+1)(\sqrt{x}+1)^2 \left[2\sqrt{2x^2+2} + (\sqrt{x}+1)\sqrt{2x+2} \right].$$

Now we shall show that $u_5(x) \geq 0, \forall x > 0$. Again in this case also we shall apply twice the argument given in Part 1. Let us consider

$$v_5(x) = (13x^3 + 26x^{5/2} + 55x^2 + 4x^{3/2} + 55x + 26\sqrt{x} + 13)^2 \\ - \left\{ 3(x+1)(\sqrt{x}+1)^2 \left[2\sqrt{2x^2+2} + (\sqrt{x}+1)\sqrt{2x+2} \right] \right\}^2 \\ = \left(\begin{array}{l} 79 + 4948x^3 + 1312x^{5/2} + 2449x^2 + 1416x^{3/2} + \\ + 1206x + 280\sqrt{x} + 2449x^4 + 1312x^{7/2} + \\ + 79x^6 + 280x^{11/2} + 1206x^5 + 1416x^{9/2} \end{array} \right) \\ - 36\sqrt{2x^2+2}\sqrt{2x+2} \left(\begin{array}{l} x^{9/2} + 5x^4 + 12x^{7/2} + \\ + 20x^3 + 26x^{5/2} + 26x^2 + \\ + 20x^{3/2} + 12x + 5\sqrt{x} + 1 \end{array} \right).$$

Let us consider again

$$v_{5a}(x) = \left(\begin{array}{l} 79 + 4948x^3 + 1312x^{5/2} + 2449x^2 + 1416x^{3/2} + \\ + 1206x + 280\sqrt{x} + 2449x^4 + 1312x^{7/2} + \\ + 79x^6 + 280x^{11/2} + 1206x^5 + 1416x^{9/2} \end{array} \right)^2 \\ - \left[36\sqrt{2x^2+2}\sqrt{2x+2} \left(\begin{array}{l} x^{9/2} + 5x^4 + 12x^{7/2} + \\ + 20x^3 + 26x^{5/2} + 26x^2 + \\ + 20x^{3/2} + 12x + 5\sqrt{x} + 1 \end{array} \right) \right]^2 \\ = (\sqrt{x}-1)^4 w_5(x),$$

where

$$w_5(x) = \left(\begin{array}{l} 1057 - 3372\sqrt{x} - 10082x + 375981x^2 + \\ + 3864616x^3 + 9785650x^4 + 1057x^{10} - 10082x^9 + \\ + 375981x^8 + 3864616x^7 + 9785650x^6 + \\ + 13943412x^5 + 1940x^{3/2} + 1462448x^{5/2} - 3372x^{19/2} + \\ + 1940x^{17/2} + 1462448x^{15/2} + 6489648x^{13/2} + \\ + 11758264x^{11/2} + 11758264x^{9/2} + 6489648x^{7/2} \end{array} \right).$$

Now we shall show that $w_5(x) > 0, \forall x > 0$. Let us consider

$$h_5(t) = w_5(t^2) = \left(\begin{array}{l} 1057t^{20} - 3372t^{19} - 10082t^{18} + 1940t^{17} + \\ + 375981t^{16} + 1462448t^{15} + 3864616t^{14} + \\ + 6489648t^{13} + 9785650t^{12} + 11758264t^{11} + \\ + 13943412t^{10} + 11758264t^9 + 9785650t^8 + \\ + 6489648t^7 + 3864616t^6 + 1462448t^5 + \\ + 375981t^4 + 1940t^3 - 10082t^2 - 3372t + 1057 \end{array} \right).$$

The polynomial equation $h_5(t) = 0$ of 20^{th} degree admits 20 solutions. All of them are convex (not written here). This means that there are no real positive solutions of the equation $h_5(t) = 0$. Thus we conclude that either $h_5(t) > 0$ or $h_5(t) < 0$, for all $t > 0$. Since $h_5(1) = 81395712$, this gives that $h_{17}(t) > 0$ for all $t > 0$, consequently, $v_{5a}(x) \geq 0$, for all $x > 0$ giving $v_5(x) \geq 0$, for all $x > 0$. Finally, we have the required result.

6. **For $\frac{10N_2+H-P_5}{10} \leq \frac{P_5+2G}{3}$:** We have to show that

$$\frac{1}{30} (13P_5 + 20G - 30N_2 - 3H) \geq 0.$$

We can write $13P_5 + 20G - 30N_2 - 3H = \frac{1}{2}b g_6(a/b)$, where

$$g_6(x) = \frac{u_6(x)}{(\sqrt{x} + 1)^2 (x + 1)},$$

with

$$u_6(x) = 26x^3 + 40x^{5/2} + 146x^2 + 56x^{3/2} + 146x + 40\sqrt{x} + 26 \\ - 15(x + 1)(\sqrt{x} + 1)^3 \sqrt{2x + 2}.$$

Now we shall show that $u_6(x) \geq 0$, $\forall x > 0$. Let us consider

$$v_6(x) = (26x^3 + 40x^{5/2} + 146x^2 + 56x^{3/2} + 146x + 40\sqrt{x} + 26)^2 \\ - \left[15(x + 1)(\sqrt{x} + 1)^3 \sqrt{2x + 2} \right]^2. \\ = 2(\sqrt{x} - 1)^4 \left(\begin{array}{l} 113x^4 + 142x^{7/2} + 436x^3 + 90x^{5/2} + \\ + 718x^2 + 90x^{3/2} + 436x + 142\sqrt{x} + 113 \end{array} \right).$$

Since $v_6(x) \geq 0$, giving $u_6(x) \geq 0$, $\forall x > 0$, hence proving the required result.

7. **For $\frac{P_6+6N_2-A}{6} \leq \frac{P_5+2G}{3}$:** We have to show that

$$\frac{1}{6} (2P_5 + 4G + A - P_4 - 6N_2) \geq 0.$$

We can write $2P_5 + 4G + A - P_4 - 6N_2 = \frac{1}{2}b g_7(a/b)$, where

$$g_7(x) = \frac{u_7(x)}{(\sqrt{x} + 1)^2},$$

with

$$u_7(x) = 5x^2 + 10x^{3/2} + 18x + 10\sqrt{x} + 5 - 3(\sqrt{x} + 1)^3 \sqrt{2x + 2}.$$

Now we shall show that $u_6(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_7(x) &= (5x^2 + 10x^{3/2} + 18x + 10\sqrt{x} + 5)^2 - \left[3(\sqrt{x} + 1)^3 \sqrt{2x + 2} \right]^2 \\ &= (\sqrt{x} - 1)^4 (7x^2 + 20x^{3/2} + 30x + 20\sqrt{x} + 7). \end{aligned}$$

Since $v_7(x) \geq 0$, giving $u_7(x) \geq 0, \forall x > 0$, hence proving the required result.

8. **For $\mathbf{P}_4 + \mathbf{P}_6 - \mathbf{S} \leq \frac{\mathbf{P}_6 + 2\mathbf{P}_4}{3}$:** We have to show that

$$\frac{1}{3} (3S - 2P_6 - P_4) \geq 0.$$

We can write $3S - 2P_6 - P_4 = \frac{1}{2}b g_8(a/b)$, where

$$g_8(x) = \frac{u_8(x)}{(x+1)(\sqrt{x}+1)^2},$$

with

$$\begin{aligned} u_8(x) &= 4x^3 + 8x^{5/2} + 12x^2 + 12x + 8\sqrt{x} + 4 \\ &\quad - 3(x+1)(\sqrt{x}+1)^2 \sqrt{2x^2 + 2}. \end{aligned}$$

Now we shall show that $u_8(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_8(x) &= (4x^3 + 8x^{5/2} + 12x^2 + 12x + 8\sqrt{x} + 4)^2 \\ &\quad - \left[3(x+1)(\sqrt{x}+1)^2 \sqrt{2x^2 + 2} \right]^2 \\ &= 2(\sqrt{x} - 1)^2 \left(\begin{aligned} &x^5 + 6x^{9/2} + 3x^4 + 12x^{7/2} + 36x^3 + \\ &+ 76x^{5/2} + 36x^2 + 12x^{3/2} + 3x + 6\sqrt{x} + 1 \end{aligned} \right). \end{aligned}$$

Since $v_8(x) \geq 0$, giving $u_8(x) \geq 0, \forall x > 0$, hence proving the required result.

9. **For $\mathbf{P}_4 + \mathbf{P}_6 - \mathbf{S} \leq \frac{\mathbf{P}_6 + 14\mathbf{N}_1}{15}$:** We have to show that

$$\frac{1}{15} (15S + 14N_1 - 14P_6 - 15P_4) \geq 0.$$

We can write $15S + 14N_1 - 14P_6 - 15P_4 = \frac{1}{2}b g_9(a/b)$, where

$$g_9(x) = \frac{u_9(x)}{(x+1)(\sqrt{x}+1)^2},$$

with

$$u_9(x) = 15(x+1)(\sqrt{x}+1)^2\sqrt{2x^2+2} - (21x^3 + 28\sqrt{x}(x-1)^2 + 99x^2 + 99x + 21).$$

Now we shall show that $u_9(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} u_9(x) &= \left[15(x+1)(\sqrt{x}+1)^2\sqrt{2x^2+2} \right]^2 \\ &\quad - (21x^3 + 28\sqrt{x}(x-1)^2 + 99x^2 + 99x + 21)^2 \\ &= (\sqrt{x}-1)^2 \left(\begin{aligned} &\sqrt{x}(34x^3 + 571x^2 + 571x + 34)(\sqrt{x}-1)^2 + \\ &+ 9x^5 + 608x^{9/2} + x^4 + 827x^{7/2} + 6710x^{5/2} + \\ &+ 827x^{3/2} + x + 608\sqrt{x} + 9 \end{aligned} \right). \end{aligned}$$

Since $v_9(x) \geq 0$, giving $u_9(x) \geq 0, \forall x > 0$, hence proving the required result.

10. **For $\mathbf{P}_4 + \mathbf{P}_6 - \mathbf{S} \leq \frac{\mathbf{S}+8\mathbf{N}_1}{9}$:** We have to show that

$$\frac{1}{9}(10S + 8N_1 - 9P_6 - 9P_4) \geq 0.$$

We can write $10S + 8N_1 - 9P_6 - 9P_4 = b g_{10}(a/b)$, where

$$g_{10}(x) = \frac{u_{10}(x)}{(x+1)(\sqrt{x}+1)^2},$$

with

$$u_{10}(x) = 5(x+1)(\sqrt{x}+1)^2\sqrt{2x^2+2} - \left[7x^3 + 10x^{5/2} + 23x^2 + 8x(\sqrt{x}-1)^2 + 23x + 10\sqrt{x} + 7 \right].$$

Now we shall show that $u_{10}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} u_{10}(x) &= \left[5(x+1)(\sqrt{x}+1)^2\sqrt{2x^2+2} \right]^2 \\ &\quad - \left[7x^3 + 10x^{5/2} + 23x^2 + 8x(\sqrt{x}-1)^2 + 23x + 10\sqrt{x} + 7 \right]^2 \\ &= (\sqrt{x}-1)^2 \left(\begin{aligned} &\sqrt{x}(6x^3 + 37x^2 + 37x + 6)(\sqrt{x}-1)^2 + \\ &+ x^5 + 56x^{9/2} + x^4 + 71x^{7/2} + 690x^{5/2} + \\ &+ 71x^{3/2} + 56\sqrt{x} + x + 1 \end{aligned} \right). \end{aligned}$$

Since $v_{10}(x) \geq 0$, giving $u_{10}(x) \geq 0, \forall x > 0$, hence proving the required result.

11. **For $\frac{P_5+2G}{3} \leq \frac{P_6+2P_4}{3}$:** We have to show that

$$\frac{1}{3} (P_6 + 2P_4 - P_5 - 2G) \geq 0.$$

We can write $P_6 + 2P_4 - P_5 - 2G = b g_{11} (a/b)$, where

$$g_{11}(x) = \frac{x(\sqrt{x} - 1)^2}{(x + 1)(x^{3/2} + 1)(\sqrt{x} + 1)}.$$

Since $g_{11}(x) \geq 0, \forall x > 0$, hence proving the required result.

12. **For $\frac{3A+G}{4} \leq \frac{P_5+9N_3}{10}$:** We have to show that

$$\frac{1}{20} (2P_5 + 18N_3 - 15A - 5G) \geq 0.$$

We can write $2P_5 + 18N_3 - 15A - 5G = \frac{1}{2}b g_{12} (a/b)$, where

$$g_{12}(x) = \frac{(\sqrt{x} - 1)^4}{(\sqrt{x} + 1)^2}.$$

Since $g_{12}(x) \geq 0, \forall x > 0$, hence proving the required result.

13. **For $\frac{P_5+9N_3}{10} \leq \frac{S+4N_2}{5}$:** We have to show that

$$\frac{1}{10} (2S + 8N_2 - P_5 - 9N_3) \geq 0.$$

We can write $2S + 8N_2 - P_5 - 9N_3 = b g_{13} (a/b)$, where

$$g_{13}(x) = \frac{u_{13}(x)}{(\sqrt{x} + 1)^2},$$

with

$$\begin{aligned} u_{13}(x) &= (\sqrt{x} + 1)^2 \left[\sqrt{2x^2 + 2} + 2\sqrt{2x + 2} (\sqrt{x} + 1) \right] \\ &\quad - (4x^2 + 9x^{3/2} + 14x + 9\sqrt{x} + 4). \end{aligned}$$

Now we shall show that $u_{13}(x) \geq 0, \forall x > 0$. In order to prove it we shall apply twice the argument given in Part 1. Let us consider

$$\begin{aligned} v_{13}(x) &= (\sqrt{x} + 1) \left[\sqrt{2x^2 + 2} + 2\sqrt{2x + 2} (\sqrt{x} + 1) \right]^2 \\ &\quad - (4x^2 + 9x^{3/2} + 14x + 9\sqrt{x} + 4)^2 \\ &= 4\sqrt{2x^2 + 2}\sqrt{2x + 2} (\sqrt{x} + 1)^5 \\ &\quad - \left(6x^4 + 16x^{7/2} + 53x^3 + 108x^{5/2} + \right. \\ &\quad \left. + 146x^2 + 108x^{3/2} + 53x + 16\sqrt{x} + 6 \right). \end{aligned}$$

Let us consider again

$$\begin{aligned}
v_{13a}(x) &= \left[4\sqrt{2x^2+2}\sqrt{2x+2}(\sqrt{x}+1)^5 \right]^2 \\
&\quad - \left(\begin{aligned} &6x^4 + 16x^{7/2} + 53x^3 + 108x^{5/2} + \\ &+ 146x^2 + 108x^{3/2} + 53x + 16\sqrt{x} + 6 \end{aligned} \right)^2 \\
&= (\sqrt{x}-1)^2 \left(\begin{aligned} &28 + 504\sqrt{x} + 3032x + 27111x^2 + 72213x^3 + \\ &+ 27111x^5 + 3032x^6 + 28x^7 + 72213x^4 + \\ &+ 10888x^{3/2} + 50366x^{5/2} + 504x^{13/2} + \\ &+ 10888x^{11/2} + 50366x^{9/2} + 81316x^{7/2} \end{aligned} \right).
\end{aligned}$$

Since $v_{13a}(x) \geq 0$, giving $v_{13}(x) \geq 0$, $\forall x > 0$. This implies that $u_{13}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

14. **For $\frac{P_5+9N_3}{10} \leq \frac{S+3N_3}{4}$:** We have to show that

$$\frac{1}{20} (5S - 3N_3 - 2P_5) \geq 0.$$

We can write $5S - 3N_3 - 2P_5 = \frac{1}{2}b g_{14}(a/b)$, where

$$g_{14}(x) = \frac{u_{14}(x)}{(\sqrt{x}+1)^2},$$

with

$$u_{14}(x) = 5\sqrt{2x^2+2}(\sqrt{x}+1)^2 - 2(3x^2 + 3x^{3/2} + 8x + 3\sqrt{x} + 3).$$

Now we shall show that $u_{14}(x) \geq 0$, $\forall x > 0$. Let us consider

$$\begin{aligned}
v_{14}(x) &= \left[5\sqrt{2x^2+2}(\sqrt{x}+1)^2 \right]^2 - \left[2(3x^2 + 3x^{3/2} + 8x + 3\sqrt{x} + 3) \right]^2 \\
&= 2(\sqrt{x}-1)^2 \left(\begin{aligned} &7x^3 + 78x^{5/2} + 185x^2 + \\ &+ 260x^{3/2} + 185x + 78\sqrt{x} + 7 \end{aligned} \right).
\end{aligned}$$

Since $v_{14}(x) \geq 0$, giving $u_{14}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

15. **For $\frac{P_5+9N_3}{10} \leq \frac{2S+H}{3}$:** We have to show that

$$\frac{1}{30} (20S + 10H - 3P_5 - 27N_3) \geq 0.$$

We can write $20S + 10H - 3P_5 - 27N_3 = b g_{15}(a/b)$, where

$$g_{15}(x) = \frac{u_8(x)}{(x+1)(\sqrt{x}+1)^2},$$

with

$$u_{15}(x) = 10\sqrt{2x^2+2}(\sqrt{x}+1)^2(x+1) - (12x^3 + 27x^{5/2} + 14x^{3/2} + 34x^2 + 34x + 27\sqrt{x} + 12).$$

Now we shall show that $u_{15}(x) \geq 0$, $\forall x > 0$. Let us consider

$$\begin{aligned} v_{15}(x) &= \left[10\sqrt{2x^2+2}(\sqrt{x}+1)^2(x+1) \right]^2 \\ &\quad - (12x^3 + 27x^{5/2} + 14x^{3/2} + 34x^2 + 34x + 27\sqrt{x} + 12)^2 \\ &= (\sqrt{x}-1)^2 \left(\begin{aligned} &56x^5 + 264x^{9/2} + 527x^4 + 1018x^{7/2} + \\ &+ 1781x^3 + 2308x^{5/2} + 1781x^2 + \\ &+ 1018x^{3/2} + 527x + 264\sqrt{x} + 56 \end{aligned} \right). \end{aligned}$$

Since $v_{15}(x) \geq 0$, giving $u_{15}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

16. **For $\frac{P_5+9N_3}{10} \leq \frac{8P_6+9H-9A}{8}$:** We have to show that

$$\frac{1}{40} (40P_6 + 45H - 45A - 4P_5 - 36N_3) \geq 0.$$

We can write $40P_6 + 45H - 45A - 4P_5 - 36N_3 = \frac{1}{2}b g_{16}(a/b)$, where

$$g_{16}(x) = \frac{(\sqrt{x}+3)(3\sqrt{x}+1)(\sqrt{x}-1)^4}{(x+1)(\sqrt{x}+1)^2}.$$

Since $g_{16}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

17. **For $\frac{8P_6+9H-9A}{8} \leq A$:** We have to show that

$$\frac{1}{8} (17A - 8P_6 - 9H) \geq 0.$$

We can write $17A - 8P_6 - 9H = \frac{1}{2}b g_{17}(a/b)$, where

$$g_{17}(x) = \frac{(x-1)^2}{(x+1)}.$$

Obviously, $g_{17}(x) > 0$, $\forall x > 0$. This proves the required result.

18. **For $\frac{14P_6+9P_3-9P_5}{14} \leq \frac{5H+4S-4P_4}{5}$:** We have to show that

$$\frac{1}{70} (45P_5 + 70H + 56S - 56P_4 - 70P_6 - 45P_3) \geq 0.$$

We can write $45P_5 + 70H + 56S - 56P_4 - 70P_6 - 45P_3 = b g_{18}(a/b)$, where

$$g_{18}(x) = \frac{u_{18}(x)}{(\sqrt{x} + 1)(x^{3/2} + 1)(x + 1)},$$

with

$$u_{18}(x) = 28\sqrt{2x^2 + 2}(x + 1)(x^{3/2} + 1)(\sqrt{x} + 1) - \left(\begin{array}{l} 25x^4 + 115x^{7/2} - 51x^3 - 69x^{5/2} + \\ + 408x^2 - 69x^{3/2} - 51x + 115\sqrt{x} + 25 \end{array} \right).$$

Now we shall show that $u_{18}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{18}(x) &= 28\sqrt{2x^2 + 2}(x + 1)(x^{3/2} + 1)(\sqrt{x} + 1) \\ &\quad - \left(\begin{array}{l} 25x^4 + 115x^{7/2} - 51x^3 - 69x^{5/2} + \\ + 408x^2 - 69x^{3/2} - 51x + 115\sqrt{x} + 25 \end{array} \right) \\ &= (\sqrt{x} - 1)^2 w_{18}(x), \end{aligned}$$

where

$$w_{18}(x) = \left(\begin{array}{l} 943x^7 - 728x^{13/2} - 8370x^6 + 8576x^{11/2} + \\ + 30935x^5 - 28454x^{9/2} - 12184x^4 + \\ + 81284x^{7/2} - 12184x^3 - 28454x^{5/2} \\ + 30935x^2 + 8576x^{3/2} - 8370x - 728\sqrt{x} + 943 \end{array} \right).$$

Now we shall show that $w_{18}(x) > 0, \forall x > 0$. Let us consider

$$h_{18}(t) = w_{18}(t^2) = \left(\begin{array}{l} 943t^{14} - 728t^{13} - 8370t^{12} + 8576t^{11} + \\ + 30935t^{10} - 28454t^9 - 12184t^8 + \\ + 81284t^7 - 12184t^6 - 28454t^5 + \\ + 30935t^4 + 8576t^3 - 8370t^2 - 728t + 943 \end{array} \right).$$

The polynomial equation $h_{18}(t) = 0$ of 14^{th} degree admits 14 solutions. Out of them 12 are convex (not written here) and two of them are real given by -1.566438336 and -0.6383909134 . These two real solutions are negative. Since we are working with $t > 0$, this means that there are no real positive solutions of the equation $h_{18}(t) = 0$. Since, $h_{18}(1) = 62720$, this gives that $h_{18}(t) > 0$, for all $t > 0$. Thus we have $w_{18}(x) > 0$, for all $x > 0$ giving $v_{18}(x) \geq 0$, for all $x > 0$. Finally, proving the required result.

19. **For $\frac{14P_6+9P_3-9P_5}{14} \leq \frac{P_5+6N_2-G}{6}$:** Equivalently, we have to show that

$$\frac{1}{42} (34P_5 + 42N_2 - 42P_6 - 27P_3 - 7G) \geq 0.$$

We can write $34P_5 + 42N_2 - 42P_6 - 27P_3 - 7G = \frac{1}{2}b g_{19}(a/b)$, where

$$g_{19}(x) = \frac{u_{19}(x)}{(\sqrt{x} + 1)(x^{3/2} + 1)(x + 1)},$$

with

$$u_{19}(x) = 21\sqrt{2x+2}(x+1)(x^{3/2}+1)(\sqrt{x}+1)^2 - \left(\begin{array}{l} 16x^4 + 64x^{7/2} + 256x^{5/2} + 256x^{3/2} + 64\sqrt{x} + \\ + 2\sqrt{x}(51x^2 + 26x + 51)(\sqrt{x}-1)^2 + 16 \end{array} \right)$$

Now we shall show that $u_{19}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{19}(x) &= \left[21\sqrt{2x+2}(x+1)(x^{3/2}+1)(\sqrt{x}+1)^2 \right]^2 \\ &\quad - \left(\begin{array}{l} 16x^4 + 64x^{7/2} + 256x^{5/2} + 256x^{3/2} + 64\sqrt{x} + \\ + 2\sqrt{x}(51x^2 + 26x + 51)(\sqrt{x}-1)^2 + 16 \end{array} \right)^2 \\ &= 2(\sqrt{x}-1)^2 w_{19}(x), \end{aligned}$$

where

$$w_{19}(x) = \left(\begin{array}{l} 313x^7 - 266x^{13/2} - 7390x^6 + 20728x^{11/2} - \\ - 25128x^5 + 41882x^{9/2} - 36915x^4 + \\ + 70000x^{7/2} - 36915x^3 + 41882x^{5/2} - \\ - 25128x^2 + 20728x^{3/2} - 7390x - 266\sqrt{x} + 313 \end{array} \right).$$

Now we shall show that $w_{19}(x) > 0, \forall x > 0$. Let us consider

$$h_{19}(t) = w_{19}(t^2) = \left(\begin{array}{l} 313t^{14} - 266t^{13} - 7390t^{12} + 20728t^{11} - 25128t^{10} + \\ + 41882t^9 - 36915t^8 + 70000t^7 - 36915t^6 + \\ + 41882t^5 - 25128t^4 + 20728t^3 - 7390t^2 - 266t + 313 \end{array} \right).$$

The polynomial equation $h_{19}(t) = 0$ of 14^{th} degree admits 14 solutions. Out of them 12 are complex (not written here) and two are real given by -5.779189781 and -0.1730346360 . These two real solutions are negative. Since we are working with $t > 0$, this means that there are no real positive solutions of the equation $h_{19}(t) = 0$. Thus we conclude that either $h_{19}(t) > 0$ or $h_{19}(t) < 0$, for all $t > 0$. In order to check it is sufficient to see for any particular value of $h_{19}(t)$, for example when $t = 1$. This gives $h_{19}(1) = 56448$, hereby proving that $h_{19}(t) > 0$ for all $t > 0$, consequently, $w_{19}(x) > 0$, for all $x > 0$ giving $v_{19}(x) \geq 0$, for all $x > 0$. Finally, proving the required result.

20. **For $\frac{P_5+6N_2-G}{6} \leq \frac{P_5+2N_2}{3}$:** Equivalently, we have to show that

$$\frac{1}{6} (P_5 - 2N_2 + G) \geq 0.$$

We can write $P_5 - 2N_2 + G = \frac{1}{2}b g_{20}(a/b)$, where

$$g_{20}(x) = \frac{u_{20}(x)}{(\sqrt{x} + 1)^2},$$

with

$$u_{20}(x) = 2(x^2 + x^{3/2} + 4x + \sqrt{x} + 1) - \sqrt{2x+2}(\sqrt{x} + 1)^3$$

Now we shall show that $u_{20}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{20}(x) &= [2(x^2 + x^{3/2} + 4x + \sqrt{x} + 1)]^2 - [\sqrt{2x+2}(\sqrt{x} + 1)^3]^2 \\ &= 2(\sqrt{x} - 1)^4(x^2 + 2x^{3/2} + 4x + 2\sqrt{x} + 1). \end{aligned}$$

Since $v_{20}(x) \geq 0$, giving $u_{20}(x) \geq 0, \forall x > 0$, hence proving the required result.

21. **For $\frac{P_5+2N_2}{3} \leq \frac{6P_6+7H-6N_2}{7}$:** Equivalently, we have to show that

$$\frac{1}{21} (18P_6 + 21H - 14N_2 - 18N_3 - 7P_5) \geq 0.$$

We can write $18P_6 + 21H - 14N_2 - 18N_3 - 7P_5 = \frac{1}{2}b g_{21}(a/b)$, where

$$g_{21}(x) = \frac{u_{21}(x)}{(x+1)(\sqrt{x} + 1)^2},$$

with

$$\begin{aligned} u_{21}(x) &= 2(5x^3 + 18x^{5/2} + 9x^2 + 48x^{3/2} + 9x + 18\sqrt{x} + 5) \\ &\quad - 7\sqrt{2x+2}(\sqrt{x} + 1)^3(x+1). \end{aligned}$$

Now we shall show that $u_{21}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{21}(x) &= [2(5x^3 + 18x^{5/2} + 9x^2 + 48x^{3/2} + 9x + 18\sqrt{x} + 5)]^2 \\ &\quad - [7\sqrt{2x+2}(\sqrt{x} + 1)^3(x+1)]^2 \\ &= 2(\sqrt{x} - 1)^4 \left(x^4 + 70x^{7/2} + 220x^3 + 210x^{5/2} + \right. \\ &\quad \left. + 510x^2 + 210x^{3/2} + 220x + 70\sqrt{x} + 1 \right). \end{aligned}$$

Since $v_{21}(x) \geq 0$, giving $u_{21}(x) \geq 0, \forall x > 0$, hence proving the required result.

22. For $\frac{5H+4S-4P_4}{5} \leq \frac{12P_5+13P_2-12P_1}{13}$: We have to show that

$$\frac{1}{65} (60P_5 + 65P_2 + 52P_4 - 60P_1 - 65H - 52S) \geq 0.$$

We can write $60P_5 + 65P_2 + 52P_4 - 60P_1 - 65H - 52S = b g_{22}(a/b)$, where

$$g_{22}(x) = \frac{u_{22}(x)}{(\sqrt{x} + 1)^2 (x^3 + 1) (x^2 + 1)},$$

with

$$u_{22}(x) = \begin{pmatrix} 60x^7 + 203x^6 - 250x^{11/2} + 190x^5 + \\ + 390x^{9/2} + 203x^4 - 760x^{7/2} + 203x^3 + \\ + 390x^{5/2} + 190x^2 - 250x^{3/2} + 203x + 60 \\ - 26\sqrt{2x^2 + 2} (x^2 + 1) (x^3 + 1) (\sqrt{x} + 1)^2 \end{pmatrix}.$$

Now we shall show that $u_{22}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{22}(x) &= \begin{pmatrix} 60x^7 + 203x^6 - 250x^{11/2} + 190x^5 + \\ + 390x^{9/2} + 203x^4 - 760x^{7/2} + 203x^3 + \\ + 390x^{5/2} + 190x^2 - 250x^{3/2} + 203x + 60 \end{pmatrix}^2 \\ &\quad - \left[26\sqrt{2x^2 + 2} (x^2 + 1) (x^3 + 1) (\sqrt{x} + 1)^2 \right]^2 \\ &= (\sqrt{x} - 1)^2 w_{22}(x), \end{aligned}$$

where

$$w_{22}(x) = \begin{pmatrix} 2248x^{13} - 912x^{25/2} + 12176x^{12} - 10144x^{23/2} + \\ + 26137x^{11} - 8506x^{21/2} + 93811x^{10} + 141228x^{19/2} + \\ + 112187x^9 - 158954x^{17/2} + 249211x^8 + \\ + 505084x^{15/2} + 137574x^7 - 471720x^{13/2} + \\ + 137574x^6 + 505084x^{11/2} + 249211x^5 - 158954x^{9/2} + \\ + 112187x^4 + 141228x^{7/2} + 93811x^3 - 8506x^{5/2} + \\ + 26137x^2 - 10144x^{3/2} + 12176x - 912\sqrt{x} + 2248 \end{pmatrix}.$$

Now we shall show that $w_{22}(x) > 0, \forall x > 0$. Let us reorganize the positive and neg-

ative terms of $w_{22}(x)$ and apply again the argument given in Part 1, by considering

$$\begin{aligned}
w_{22a}(x) &= \left(\begin{aligned} &2248x^{13} + 12176x^{12} + 26137x^{11} + 93811x^{10} + 141228x^{19/2} + \\ &+ 112187x^9 + 249211x^8 + 505084x^{15/2} + 137574x^7 + \\ &+ 137574x^6 + 505084x^{11/2} + 249211x^5 + 112187x^4 + \\ &+ 141228x^{7/2} + 93811x^3 + 26137x^2 + 12176x + 2248 \end{aligned} \right)^2 \\
&\quad - \left(\begin{aligned} &912x^{25/2} + 10144x^{23/2} + 8506x^{21/2} + \\ &+ 158954x^{17/2} + 471720x^{13/2} + 158954x^{9/2} + \\ &+ 8506x^{5/2} + 10144x^{3/2} + 912\sqrt{x} \end{aligned} \right)^2 \\
&= \left(\begin{aligned} &5053504 + 53911552x + 939846800x^3 + 247264272x^2 + \\ &+ 8394022562x^5 + 3439184256x^{9/2} + 9653410136x^{11/2} + \\ &+ 54425545136x^7 + 3299451265x^4 + 634961088x^{7/2} + \\ &+ 177455818032x^{35/2} + 388807633696x^{31/2} + 60361509952x^{37/2} + \\ &+ 54425545136x^{19} + 18127427183x^{20} + 60361509952x^{15/2} + \\ &+ 61435331831x^8 + 38797285384x^{13/2} + 399552515016x^{11} + \\ &+ 330074264536x^{23/2} + 179224041264x^{19/2} + 204575899694x^9 + \\ &+ 388807633696x^{21/2} + 177455818032x^{17/2} + 169865767282x^{12} \end{aligned} \right) \\
&\quad + \left(\begin{aligned} &131742209367x^{10} + 483216926288x^{13} + 448903200968x^{25/2} + \\ &+ 38797285384x^{39/2} + 9653410136x^{41/2} + 8394022562x^{21} + \\ &+ 3299451265x^{22} + 939846800x^{23} + 247264272x^{24} + \\ &+ 3439184256x^{43/2} + 634961088x^{5/2} + 53911552x^{25} + \\ &+ 5053504x^{26} + 18127427183x^6 + 169865767282x^{14} + \\ &+ 399552515016x^{15} + 131742209367x^{16} + \\ &+ 330074264536x^{29/2} + 204575899694x^{17} + \\ &+ 61435331831x^{18} + 448903200968x^{27/2} + 179224041264x^{33/2} \end{aligned} \right).
\end{aligned}$$

Thus the positivity of $w_{22a}(x)$ proves that $w_{22}(x) > 0$, $\forall x > 0$. This completes the proof of the result.

23. **For $\frac{5H+4S-4P_4}{5} \leq \frac{4H+5S-5G}{4}$:** We have to show that

$$\frac{1}{20} (16P_4 + 9S - 25G) \geq 0.$$

We can write $16P_4 + 9S - 25G = \frac{1}{2}b g_{23}(a/b)$, where

$$g_{23}(x) = \frac{u_{23}(x)}{(\sqrt{x} + 1)^2},$$

with

$$u_{23}(x) = 9(\sqrt{x} + 1)^2 \sqrt{2x^2 + 2} - \sqrt{x} \left[36(x + 1) + 14(\sqrt{x} - 1)^2 \right].$$

Now we shall show that $u_{23}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{23}(x) &= \left\{ 9 (\sqrt{x} + 1)^2 \sqrt{2x^2 + 2} \right\}^2 \\ &\quad - \left\{ \sqrt{x} \left[36(x + 1) + 14(\sqrt{x} - 1)^2 \right] \right\}^2 \\ &= 2 (\sqrt{x} - 1)^2 \left(\begin{aligned} &81x^3 + 486x^{5/2} + 127x^2 + \\ &+ 1492x^{3/2} + 127x + 486\sqrt{x} + 81 \end{aligned} \right). \end{aligned}$$

Since $v_{23}(x) \geq 0$, giving $u_{23}(x) \geq 0, \forall x > 0$, hence proving the required result.

24. **For $\frac{7H+6P_6-6N_3}{7} \leq \frac{12P_5+13P_2-12P_1}{13}$:** We have to show that

$$\frac{1}{91} (84P_5 + 91P_2 + 78N_3 - 84P_1 - 91H - 78P_6) \geq 0.$$

We can write $84P_5 + 91P_2 + 78N_3 - 84P_1 - 91H - 78P_6 = b g_{24}(a/b)$, where

$$g_{24}(x) = \frac{u_{24}(x)}{(\sqrt{x} + 1)^2 (x^3 + 1)(x + 1)},$$

where

$$u_{24}(x) = (\sqrt{x} - 1)^2 \left(\begin{aligned} &7(x^5 + 2x^4 + 2x + 1)(\sqrt{x} - 1)^2 + \\ &+ 25x^6 + 16x^5 + 55x^4 + 322x^{7/2} + \\ &+ 620x^3 + 322x^{5/2} + 55x^2 + 16x + 25 \end{aligned} \right).$$

Since $u_{24}(x) \geq 0, \forall x > 0$, this gives $u_{24}(x) \geq 0, \forall x > 0$, thereby proving the required result.

25. **For $\frac{7H+6P_6-6N_3}{7} \leq \frac{4H+5S-5G}{4}$:** We have to show that

$$\frac{1}{28} (35S + 24N_3 - 35G - 24P_6) \geq 0.$$

We can write $35S + 24N_3 - 35G - 24P_6 = \frac{1}{2}b g_{25}(a/b)$, where

$$g_{25}(x) = \frac{u_{25}(x)}{(x + 1)},$$

with

$$\begin{aligned} u_{25}(x) &= 35(x + 1) \sqrt{2x^2 + 2} \\ &\quad - \left[32x^2 + 38x^{3/2} + 16\sqrt{x}(\sqrt{x} - 1)^2 + 38\sqrt{x} + 32 \right]. \end{aligned}$$

Now we shall show that $u_{25}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{25}(x) &= \left[35(x+1)\sqrt{2x^2+2} \right]^2 \\ &\quad - \left[32x^2 + 38x^{3/2} + 16\sqrt{x}(\sqrt{x}-1)^2 + 38\sqrt{x} + 32 \right]^2 \\ &= 2(\sqrt{x}-1)^2 \left(\begin{array}{l} 562x^3 + 151(x^2+1)(\sqrt{x}-1)^2 + \\ + 548x^2 + 1700x^{3/2} + 548x + 562 \end{array} \right). \end{aligned}$$

Since $v_{25}(x) \geq 0$, giving $u_{25}(x) \geq 0, \forall x > 0$, hence proving the required result.

26. **For $\frac{7H+6P_6-6N_3}{7} \leq S$:** We have to show that

$$\frac{1}{7}(7S + 6N_3 - 7H - 6P_6) \geq 0.$$

We can write $7S + 6N_3 - 7H - 6P_6 = \frac{1}{2}b g_{26}(a/b)$, where

$$g_{26}(x) = \frac{u_{26}(x)}{(x+1)},$$

with

$$u_{26}(x) = 7(x+1)\sqrt{2x^2+2} - (-8x^2 + 4x^{3/2} - 20x + 4\sqrt{x} - 8).$$

Now we shall show that $u_{26}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{26}(x) &= \left[7(x+1)\sqrt{2x^2+2} \right]^2 - (-8x^2 + 4x^{3/2} - 20x + 4\sqrt{x} - 8)^2 \\ &= 2(\sqrt{x}-1)^2 \left(\begin{array}{l} 17x^3 + 66x^{5/2} + 45x^2 + \\ + 136x^{3/2} + 45x + 66\sqrt{x} + 17 \end{array} \right). \end{aligned}$$

Since $v_{26}(x) \geq 0$, giving $u_{26}(x) \geq 0, \forall x > 0$, hence proving the required result.

27. **For $\frac{P_6+2N_1-G}{2} \leq \frac{2P_4+5P_6}{7}$:** We have to show that

$$\frac{1}{14}(4P_5 + 3P_6 + 7G - 14N_1) \geq 0.$$

We can write $4P_5 + 3P_6 + 7G - 14N_1 = \frac{1}{2}b g_{27}(a/b)$, where

$$g_{27}(x) = \frac{(\sqrt{x}-1)^2(7x^2 + 12x^{3/2} + 26x + 12\sqrt{x} + 7)}{(x+1)(\sqrt{x}+1)^2},$$

Since $g_{27}(x) \geq 0, \forall x > 0$, hence proving the required result.

28. **For $\frac{P_6+2N_1-G}{2} \leq \frac{4N_2+5P_6}{9}$:** We have to show that

$$\frac{1}{18} (18N_1 - P_6 - 9G - 8N_2) \geq 0.$$

We can write $18N_1 - P_6 - 9G - 8N_2 = \frac{1}{2}b g_{28}(a/b)$, where

$$g_{28}(x) = \frac{u_{28}(x)}{(x+1)},$$

with

$$u_{28}(x) = 7x^2 + 18x + 7 - 4\sqrt{2x+2}(\sqrt{x}+1)(x+1).$$

Now we shall show that $u_{28}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{28}(x) &= (7x^2 + 18x + 7)^2 - \left[4\sqrt{2x+2}(\sqrt{x}+1)(x+1)\right]^2 \\ &= (\sqrt{x}-1)^4 (17x^2 + 4x^{3/2} + 38x + 4\sqrt{x} + 17). \end{aligned}$$

Since $v_{28}(x) \geq 0$, giving $u_{28}(x) \geq 0, \forall x > 0$, hence proving the required result.

29. **For $A + S - G \leq \frac{12P_5+7P_2-12P_3}{7}$:** We have to show that

$$\frac{1}{7} (12P_5 + 7P_2 + 7G - 12P_3 - 7A - 7S) \geq 0.$$

We can write $12P_5 + 7P_2 + 7G - 12P_3 - 7A - 7S = \frac{1}{2}b g_{29}(a/b)$, where

$$g_{29}(x) = \frac{u_{30}(x)}{(\sqrt{x}+1)(x^{3/2}+1)(x^2+1)},$$

with

$$\begin{aligned} u_{29}(x) &= \left(\frac{8x^5 + (9x^4 + 48x^3 + 3x^2 + 48x + 9)(\sqrt{x}-1)^2}{x^{9/2} + 12x^4 + 35x^3 + 35x^2 + 12x + \sqrt{x} + 8} \right) \\ &\quad - 7\sqrt{2x^2+2}(x^{3/2}+1)(x^2+1)(\sqrt{x}+1). \end{aligned}$$

Now we shall show that $u_{29}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{29}(x) &= \left(\frac{8x^5 + (9x^4 + 48x^3 + 3x^2 + 48x + 9)(\sqrt{x}-1)^2}{x^{9/2} + 12x^4 + 35x^3 + 35x^2 + 12x + \sqrt{x} + 8} \right)^2 \\ &\quad - \left[7\sqrt{2x^2+2}(x^{3/2}+1)(x^2+1)(\sqrt{x}+1) \right]^2 \\ &= (\sqrt{x}-1)^4 \left(\begin{aligned} &186x^8 + 5(x^7+1)(\sqrt{x}-1)^2 + 1346x^7 + \\ &+ 422x^{13/2} + 3614x^6 + 178x^{11/2} + 3185x^5 + \\ &+ 1394x^{9/2} + 8022x^4 + 1394x^{7/2} + 3185x^3 + \\ &+ 178x^{5/2} + 3614x^2 + 422x^{3/2} + 1346x + 186 \end{aligned} \right). \end{aligned}$$

Since $v_{29}(x) \geq 0$, giving $u_{29}(x) \geq 0, \forall x > 0$, hence proving the required result.

30. **For $\frac{6P_5+5G-6H}{5} \leq \frac{12P_5+7P_2-12P_3}{7}$:** We have to show that

$$\frac{1}{35} (18P_5 + 35P_2 + 42H - 60P_3 - 35G) \geq 0.$$

We can write $18P_5 + 35P_2 + 42H - 60P_3 - 35G = b g_{30} (a/b)$, where

$$g_{30}(x) = \frac{u_{30}(x)}{(\sqrt{x} + 1)(x^2 + 1)(x^{3/2} + 1)(x + 1)},$$

where

$$u_{30}(x) = (\sqrt{x} - 1)^4 \left(\begin{array}{l} 18x^4 + 19x^{7/2} + 64x^3 + 124x^{5/2} + \\ + 176x^2 + 124x^{3/2} + 64x + 19\sqrt{x} + 18 \end{array} \right).$$

Since $u_{30}(x) \geq 0, \forall x > 0$, this gives $u_{30}(x) \geq 0, \forall x > 0$, thereby proving the required result.

31. **For $2A + G - 2P_4 \leq \frac{12P_5+7P_2-12P_3}{7}$:** We have to show that

$$\frac{1}{7} (12P_5 + 7P_2 + 14P_4 - 12P_3 - 14A - 7G) \geq 0.$$

We can write $12P_5 + 7P_2 + 14P_4 - 12P_3 - 14A - 7G = b g_{31} (a/b)$, where

$$g_{31}(x) = \frac{u_{32}(x)}{(\sqrt{x} + 1)(x^2 + 1)(x^{3/2} + 1)},$$

where

$$u_{31}(x) = (\sqrt{x} - 1)^4 \left(\begin{array}{l} 2x^3 + 3(x^2 + 1)(\sqrt{x} - 1)^2 + \\ + 16x^2 + 21x^{3/2} + 16x + 2 \end{array} \right).$$

Since $u_{31}(x) \geq 0, \forall x > 0$, this gives $u_{31}(x) \geq 0, \forall x > 0$, thereby proving the required result.

32. **For $A + S - G \leq 6S + P_2 - 6A$:** We have to show that $5S + P_2 + G - 7A \geq 0$.
We can write $5S + P_2 + G - 7A = \frac{1}{2}b g_{32} (a/b)$, where

$$g_{32}(x) = \frac{u_{32}(x)}{(x^2 + 1)},$$

with

$$u_{32}(x) = 5\sqrt{2x^2 + 2}(x^2 + 1) - (7x^3 - 2x^{5/2} + 5x^2 - 2\sqrt{x} + 5x + 7).$$

Now we shall show that $u_{32}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{32}(x) &= \left[5\sqrt{2x^2+2} (x^2+1) \right]^2 - (7x^3 - 2x^{5/2} + 5x^2 + 5x - 2\sqrt{x} + 7)^2 \\ &= (\sqrt{x}-1)^4 \left(\begin{array}{l} x^4 + 32x^{7/2} + 48x^3 + 5x^{3/2} (\sqrt{x}-1)^2 + \\ + 19x^{5/2} + 19x^{3/2} + 48x + 32\sqrt{x} + 1 \end{array} \right). \end{aligned}$$

Since $v_{32}(x) \geq 0$, giving $u_{32}(x) \geq 0, \forall x > 0$, hence proving the required result.

33. **For $\frac{6P_5+5G-6H}{5} \leq 6S + P_2 - 6A$:** We have to show that

$$\frac{1}{5} (30S + 5P_2 + 6H - 30A - 6P_5 - 5G) \geq 0.$$

We can write $30S + 5P_2 + 6H - 30A - 6P_5 - 5G = b g_{33} (a/b)$, where

$$g_{33}(x) = \frac{u_{33}(x)}{(\sqrt{x}+1)^2 (x^2+1) (x+1)},$$

with

$$\begin{aligned} u_{33}(x) &= 15\sqrt{2x^2+2} (x^2+1) (\sqrt{x}+1)^2 (x+1) \\ &\quad - \left(\begin{array}{l} 21x^5 + 35x^{9/2} + 56x^4 + 36x^{7/2} + 67x^3 + \\ + 50x^{5/2} + 67x^2 + 36x^{3/2} + 56x + 35\sqrt{x} + 21 \end{array} \right). \end{aligned}$$

Now we shall show that $u_{33}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{33}(x) &= \left[15\sqrt{2x^2+2} (x^2+1) (\sqrt{x}+1)^2 (x+1) \right]^2 \\ &\quad - \left(\begin{array}{l} 21x^5 + 35x^{9/2} + 56x^4 + 36x^{7/2} + 67x^3 + \\ + 50x^{5/2} + 67x^2 + 36x^{3/2} + 56x + 35\sqrt{x} + 21 \end{array} \right)^2 \\ &= (\sqrt{x}-1)^4 \left(\begin{array}{l} 9x^8 + 366x^{15/2} + 1433x^7 + 3540x^{13/2} + 6197x^6 + \\ + 8892x^{11/2} + 10399x^5 + 10866x^{9/2} + 10676x^4 + \\ + 10866x^{7/2} + 10399x^3 + 8892x^{5/2} + 6197x^2 + \\ + 3540x^{3/2} + 1433x + 366\sqrt{x} + 9 \end{array} \right). \end{aligned}$$

Since $v_{33}(x) \geq 0$, giving $u_{33}(x) \geq 0, \forall x > 0$, hence proving the required result.

34. **For $2A + G - 2P_4 \leq 6S + P_2 - 6A$:** We have to show that

$$6S + P_2 + 2P_4 - G - 8A \geq 0.$$

We can write $6S + P_2 + 2P_4 - G - 8A = b g_{34} (a/b)$, where

$$g_{34}(x) = \frac{u_{34}(x)}{(x^2+1) (\sqrt{x}+1)^2},$$

with

$$u_{34}(x) = 3\sqrt{2x^2 + 2} (x^2 + 1) (\sqrt{x} + 1)^2 - \left(\begin{array}{l} 4x^4 + 9x^{7/2} + x^3 + 7x^{5/2} + \\ + 6x^2 + 7x^{3/2} + x + 9\sqrt{x} + 4 \end{array} \right).$$

Now we shall show that $u_{34}(x) \geq 0$, $\forall x > 0$. Let us consider

$$\begin{aligned} v_{34}(x) &= \left[3\sqrt{2x^2 + 2} (x^2 + 1) (\sqrt{x} + 1)^2 \right]^2 \\ &\quad - \left(\begin{array}{l} 4x^4 + 9x^{7/2} + x^3 + 7x^{5/2} + \\ + 6x^2 + 7x^{3/2} + x + 9\sqrt{x} + 4 \end{array} \right)^2 \\ &= (\sqrt{x} - 1)^4 \left(\begin{array}{l} 2x^6 + 8x^{11/2} + 39x^5 + 114x^{9/2} + \\ + 149x^4 + 98x^{7/2} + 44x^3 + 98x^{5/2} + \\ + 149x^2 + 114x^{3/2} + 39x + 8\sqrt{x} + 2 \end{array} \right). \end{aligned}$$

Since $v_{34}(x) \geq 0$, giving $u_{34}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

35. **For $6S + P_2 - 6A \leq \frac{16P_5 + 9P_1 - 16P_2}{9}$:** We have to show that

$$\frac{1}{9} (16P_5 + 9P_1 + 54A - 25P_2 - 54S) \geq 0.$$

We can write $16P_5 + 9P_1 + 54A - 25P_2 - 54S = b g_{35} (a/b)$, where

$$g_{35}(x) = \frac{u_{35}(x)}{(\sqrt{x} + 1)^2 (x^3 + 1) (x^2 + 1)},$$

with

$$u_{35}(x) = 27\sqrt{2x^2 + 2} (x^2 + 1) (x^3 + 1) (\sqrt{x} + 1)^2 - \left(\begin{array}{l} 43x^7 + 54x^{13/2} + 70x^6 + 22x^{11/2} + 45x^5 + \\ + 4x^{9/2} + 122x^4 + 144x^{7/2} + 122x^3 + 4x^{5/2} + \\ + 45x^2 + 22x^{3/2} + 70x + 54\sqrt{x} + 43 \end{array} \right).$$

Now we shall show that $u_{35}(x) \geq 0$, $\forall x > 0$. Let us consider

$$\begin{aligned} v_{35}(x) &= \left[27\sqrt{2x^2 + 2} (x^2 + 1) (x^3 + 1) (\sqrt{x} + 1)^2 \right]^2 \\ &\quad - \left(\begin{array}{l} 43x^7 + 54x^{13/2} + 70x^6 + 22x^{11/2} + 45x^5 + \\ + 4x^{9/2} + 122x^4 + 144x^{7/2} + 122x^3 + 4x^{5/2} + \\ + 45x^2 + 22x^{3/2} + 70x + 54\sqrt{x} + 43 \end{array} \right)^2 \\ &= (\sqrt{x} - 1)^4 \left(\begin{array}{l} 64x^{12} + 327(x^{10} + 1)(x - 1)^2 + 376x^{23/2} + 312x^{21/2} + \\ + 11272x^{10} + 32320x^{19/2} + 50136x^9 + 51648x^{17/2} + \\ + 42538x^8 + 38736x^{15/2} + 56978x^7 + 88128x^{13/2} + \\ + 105160x^6 + 88128x^{11/2} + 56978x^5 + 38736x^{9/2} + \\ + 42538x^4 + 51648x^{7/2} + 50136x^3 + 32320x^{5/2} + \\ + 11272x^2 + 312x^{3/2} + 376\sqrt{x} + 64 \end{array} \right). \end{aligned}$$

Since $v_{35}(x) \geq 0$, giving $u_{35}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

36. **For $\frac{12P_5+7P_2-12P_3}{7} \leq \frac{16P_5+9P_1-16P_2}{9}$:** We have to show that

$$\frac{1}{63} (4P_5 + 63P_1 + 108P_3 - 175P_2) \geq 0.$$

We can write $4P_5 + 63P_1 + 108P_3 - 175P_2 = b g_{36}(a/b)$, where

$$g_{36}(x) = \frac{u_{36}(x)}{(\sqrt{x} + 1)(x^2 + 1)(x^{3/2} + 1)(x^3 + 1)},$$

where

$$u_{36}(x) = (\sqrt{x} - 1)^4 \left(\begin{array}{l} 4x^6 + 12x^{11/2} + 32x^5 + 168x^{9/2} + \\ + 473x^4 + 705x^{7/2} + 764x^3 + 705x^{5/2} + \\ + 473x^2 + 168x^{3/2} + 32x + 12\sqrt{x} + 4 \end{array} \right).$$

Since $u_{36}(x) \geq 0$, $\forall x > 0$, this gives $u_{36}(x) \geq 0$, $\forall x > 0$, thereby proving the required result.

37. **For $\frac{P_6+3P_1}{4} \leq N_1$:** We have to show that

$$\frac{1}{4} (4N_1 - P_6 - 3P_1) \geq 0.$$

We can write $4N_1 - P_6 - 3P_1 = b g_{37}(a/b)$, where

$$g_{37}(x) = \frac{\sqrt{x}(\sqrt{x} - 1)^2(2x + 3\sqrt{x} + 2)}{(x - 1)^2 + x}.$$

Since $g_{37}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

38. **For $N_1 \leq \frac{P_6+3G}{4}$:** We have to show that

$$\frac{1}{4} (P_6 + 3G - 4N_1) \geq 0.$$

We can write $P_6 + 3G - 4N_1 = b g_{38}(a/b)$, where

$$g_{38}(x) = \frac{\sqrt{x}(\sqrt{x} - 1)^2}{x + 1}.$$

Since $g_{38}(x) \geq 0$, $\forall x > 0$, hence proving the required result.

39. **For $\frac{\mathbf{P}_6+3\mathbf{G}}{4} \leq \frac{\mathbf{S}+5\mathbf{N}_1}{6}$:** Equivalently, we have to show that

$$\frac{1}{6} (2S + 10N_1 - 3P_6 - 9G) \geq 0.$$

We can write $2S + 10N_1 - 3P_6 - 9G = \frac{1}{2}b g_{39}(a/b)$, where

$$g_{39}(x) = \frac{u_{39}(x)}{x+1},$$

with

$$\begin{aligned} u_{39}(x) &= 2\sqrt{2x^2+2}(x+1) \\ &\quad - \left(x^2 + 3x^{3/2} + 5\sqrt{x}(\sqrt{x}-1)^2 + 3\sqrt{x}+1 \right). \end{aligned}$$

Now we shall show that $u_{39}(x) \geq 0, \forall x > 0$. Let us consider

$$\begin{aligned} v_{39}(x) &= \left[2\sqrt{2x^2+2}(x+1) \right]^2 \\ &\quad - \left(x^2 + 3x^{3/2} + 5\sqrt{x}(\sqrt{x}-1)^2 + 3\sqrt{x}+1 \right)^2 \\ &= (\sqrt{x}-1)^4 \left[7x^2 + 11\sqrt{x}(\sqrt{x}-1)^2 + \sqrt{x}(x+1) + 7 \right]. \end{aligned}$$

Since $v_{39}(x) \geq 0$, giving $u_{39}(x) \geq 0, \forall x > 0$, hence proving the required result.

40. **For $\frac{4\mathbf{P}_5+9\mathbf{P}_1}{13} \leq \mathbf{N}_3$:** We have to show that

$$\frac{1}{13} (13N_3 - 4P_5 - 9P_1) \geq 0.$$

We can write $13N_3 - 4P_5 - 9P_1 = \frac{1}{3}b g_{40}(a/b)$, where

$$g_{40}(x) = \frac{(\sqrt{x}-1)^2 \left(x^4 + 41x^{7/2} + 82x^3 + 108x^{5/2} + 108x^2 + 108x^{3/2} + 82x + 41\sqrt{x}+1 \right)}{(\sqrt{x}+1)^2(x^3+1)}.$$

Since $g_{40}(x) \geq 0, \forall x > 0$, hence proving the required result.

41. **For $\frac{\mathbf{P}_5+5\mathbf{N}_1}{6} \leq \mathbf{A}$:** We have to show that

$$\frac{1}{6} (6A - P_5 - 5N_1) \geq 0.$$

We can write $6A - P_5 - 5N_1 = \frac{1}{4}b g_{41}(a/b)$, where

$$g_{41}(x) = \frac{(\sqrt{x}-1)^2(\sqrt{x}+3)(3\sqrt{x}+1)}{(\sqrt{x}+1)^2}.$$

Since $g_{41}(x) \geq 0, \forall x > 0$, hence proving the required result.

42. For $\frac{2P_5+7P_2}{9} \leq N_1$: We have to show that

$$\frac{1}{9}(9N_1 - 2P_5 - 7P_2) \geq 0.$$

We can write $9N_1 - 2P_5 - 7P_2 = \frac{1}{4}b g_{42}(a/b)$, where

$$g_{42}(x) = \frac{(\sqrt{x} - 1)^2 \left(x^3 + 38x^{5/2} + 85x^2 + 112x^{3/2} + 85x + 38\sqrt{x} + 1 \right)}{(\sqrt{x} + 1)^2 (x^2 + 1)}.$$

Since $g_{42}(x) \geq 0, \forall x > 0$, hence proving the required result.

□

Remark 2.1. The results studied above in 1-42 parts can equivalently be written as:

- | | |
|--|--|
| 1. $D_{P_6S} \leq \frac{1}{4}(3D_{SP_4} + 21D_{GP_4}).$ | 15. $D_{SN_3} \geq \frac{1}{20}(7D_{N_3H} + 3D_{P_5H}).$ |
| 2. $D_{SN_2} \leq \frac{5}{3}D_{N_2G}.$ | 16. $D_{AH} \leq \frac{1}{45}(36D_{P_6N_3} + 4D_{P_6P_5}).$ |
| 3. $D_{P_6N_2} \leq 3D_{N_2G}.$ | 17. $D_{P_6A} \leq \frac{9}{8}D_{AH}.$ |
| 4. $D_{N_2P_4} \leq \frac{1}{10}(10D_{P_6S} + D_{P_5H}).$ | 18. $D_{P_6H} \leq \frac{1}{70}(56D_{SP_4} + 45D_{P_5P_3}).$ |
| 5. $D_{N_2P_4} \leq \frac{1}{5}(6D_{P_6S} + D_{AN_2}).$ | 19. $D_{P_6N_2} \leq \frac{1}{42}(27D_{P_5P_3} + 7D_{P_5G}).$ |
| 6. $D_{N_2G} \leq \frac{1}{20}(10D_{P_5N_2} + 3D_{P_5H}).$ | 20. $D_{N_2G} \leq D_{P_5N_2}.$ |
| 7. $D_{N_2G} \leq \frac{1}{4}(2D_{P_5N_2} + D_{AP_4}).$ | 21. $D_{P_6N_3} \geq \frac{1}{18}(14D_{N_2H} + 7D_{P_5H}).$ |
| 8. $D_{P_6S} \leq \frac{1}{2}D_{SP_4}.$ | 22. $D_{P_5P_1} \geq \frac{1}{60}(65D_{HP_2} + 52D_{SP_4}).$ |
| 9. $D_{P_6N_1} \leq \frac{15}{14}D_{SP_4}.$ | 23. $D_{SP_4} \leq \frac{25}{16}D_{SG}.$ |
| 10. $D_{P_6S} \leq \frac{1}{9}(D_{SP_4} + 8D_{N_1P_4}).$ | 24. $D_{P_5P_1} \leq \frac{1}{84}(91D_{HP_2} + 78D_{P_6N_3}).$ |
| 11. $D_{GP_4} \leq \frac{1}{2}D_{P_6P_5}.$ | 25. $D_{P_6N_3} \leq \frac{35}{24}D_{SG}.$ |
| 12. $D_{AN_3} \leq \frac{1}{13}(2D_{P_5A} + 5D_{N_3G}).$ | 26. $D_{P_6N_3} \leq \frac{7}{6}D_{SH}.$ |
| 13. $D_{P_5N_2} \leq 2D_{SN_3} + 7D_{N_2N_3}.$ | 27. $D_{N_1G} \leq \frac{1}{7}(4D_{P_5N_1} + 3D_{P_6N_1}).$ |
| 14. $D_{P_5N_3} \leq \frac{5}{2}D_{SN_3}.$ | 28. $D_{N_1G} \geq \frac{1}{9}(D_{P_6N_1} + 8D_{N_2N_1}).$ |

$$\begin{aligned}
29. \quad D_{P_5 P_3} &\geq \frac{7}{12} (D_{AP_2} + D_{SG}). & 36. \quad D_{P_2 P_1} &\leq \frac{1}{63} (4D_{P_5 P_2} + 108D_{P_3 P_2}). \\
30. \quad D_{GP_2} &\leq \frac{1}{35} (18D_{P_5 P_3} + 42D_{HP_3}). & 37. \quad D_{P_6 N_1} &\leq 3D_{N_1 P_1}. \\
31. \quad D_{P_5 P_3} &\geq \frac{1}{12} (7D_{GP_2} + 14D_{AP_4}). & 38. \quad D_{N_1 G} &\leq \frac{1}{3} D_{P_6 N_1}. \\
32. \quad D_{AG} + D_{AP_2} &\leq 5D_{SA}. & 39. \quad D_{P_6 N_1} &\leq \frac{1}{3} (2D_{SG} + 7D_{N_1 G}). \\
33. \quad D_{SA} &\geq \frac{1}{30} (5D_{GP_2} + 6D_{P_5 H}). & 40. \quad D_{P_5 N_3} &\leq \frac{9}{4} D_{N_3 P_1}. \\
34. \quad D_{SA} &\geq \frac{1}{6} (2D_{AP_4} + D_{GP_2}). & 41. \quad D_{P_5 A} &\leq 5D_{AN_1}. \\
35. \quad D_{P_5 P_2} &\geq \frac{1}{16} (9D_{P_2 P_1} + 54D_{SA}). & 42. \quad D_{P_5 N_1} &\leq \frac{7}{2} D_{N_1 P_2}.
\end{aligned}$$

3 Connections with Divergence Measures

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \dots, p_n) \left| p_i > 0, \sum_{i=1}^n p_i = 1 \right. \right\}, \quad n \geq 2,$$

be the set of all complete finite discrete probability distributions. For all $P, Q \in \Gamma_n$, the author [10, 11] proved the following inequalities:

$$\begin{aligned}
\frac{1}{2} D_{AH}(P||Q) &\leq I(P||Q) \leq 4D_{N_2 N_1}(P||Q) \leq \frac{4}{3} D_{N_2 G}(P||Q) \\
&\leq D_{AG}(P||Q) \leq 4 D_{AN_2}(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
I(P||Q) &= \frac{1}{2} \left[\sum_{i=1}^n p_i \ln \left(\frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^n q_i \ln \left(\frac{2q_i}{p_i + q_i} \right) \right], \\
J(P||Q) &= \sum_{i=1}^n (p_i - q_i) \ln \left(\frac{p_i}{q_i} \right)
\end{aligned}$$

and

$$T(P||Q) = \sum_{i=1}^n \left(\frac{p_i + q_i}{2} \right) \ln \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right).$$

The measures $I(P||Q)$, $J(P||Q)$ and $T(P||Q)$ are the respectively, the well-know *Jensen-Shannon divergence*, *J-divergence* and *arithmetic and geometric mean divergence*. Moreover, $D_{AH}(P||Q) = \frac{1}{2}\Delta(P||Q)$ and $D_{AG}(P||Q) = h(P||Q)$, where $\Delta(P||Q)$

and $h(P||Q)$ are the well-known *triangular's* and *Hellinger's discriminations* respectively. These three measures satisfy the following equality:

$$J(P||Q) = 4 [I(P||Q) + T(P||Q)]. \quad (17)$$

Instead of $D_{(\cdot)}(a, b)$ as given in Section 1, here we are working with the probability distributions P and Q , and writing $D_{(\cdot)}(a, b)$ as $D_{(\cdot)}(P||Q)$. Recently, author [13] proved the following inequalities:

- (i) $\frac{2}{3}D_{AP_4} \leq I$;
- (ii) $\frac{2}{3}D_{P_5G} \leq \frac{1}{8}J$;
- (iii) $D_{P_5A} \leq \frac{1}{2}T$.

Theorem 3.1. *The following inequalities hold:*

$$N_3 \leq \frac{I + 4N_1}{4} \leq N_2 \leq A \leq \frac{2P_4 + 3I}{2} \leq \begin{cases} P_5 \leq \frac{T+2A}{2} \leq \frac{3J+16G}{16} \\ S \end{cases}. \quad (18)$$

Proof. In view of (i)-(iii), we shall prove only the necessary parts.

1. **For $N_3 \leq \frac{I+4N_1}{4}$:** We can write it as $4D_{N_3N_1} \leq I$. Since $D_{N_3N_1} = \frac{1}{6}D_{AG}$, we have to show that $\frac{2}{3}D_{AG} \leq I$. According to (8), $D_{AG} \leq D_{AP_4}$. This together with (i) proves the requires result.
2. **For $\frac{2P_4+3I}{2} \leq P_5$:** We can write it as $I \leq \frac{2}{3}D_{P_5P_4}$. Since, $D_{P_5P_4} = 2D_{AG}$ and $I \leq 4D_{N_2N_1} \leq D_{AG}$. This implies that $I \leq D_{AG} = \frac{1}{2}D_{P_5P_4} \leq \frac{2}{3}D_{P_5P_4}$. This gives the requires result.
3. **For $\frac{T+2A}{2} \leq \frac{3J+16G}{16}$:** We have to show that $\frac{3}{16}J + G - \frac{1}{2}T - A \geq 0$, i.e., $A - G \leq \frac{3}{16}J - \frac{1}{2}T$. Since $J = 4(I + T)$, $D_{AG} \leq I$ and $\frac{1}{8}J \leq T$, combining we get the required result.
4. **For $\frac{2P_4+3I}{2} \leq S$:** We can write $I \leq \frac{2}{3}D_{SP_4}$. We know that $I \leq 4D_{N_2N_1}$. We shall prove that $I \leq 4D_{N_2N_1} \leq \frac{2}{3}D_{SP_4}$. In order to show this we need to show that $4D_{N_2N_1} \leq \frac{2}{3}D_{SP_4}$, i.e.,

$$\frac{1}{3}(2S + 12N_1 - 2P_4 - 12N_2) \geq 0.$$

We can write $2S + 12N_1 - 2P_4 - 12N_2 = \sum_{i=1}^n q_i g_{43}(p_i/q_i)$, where

$$g_{43}(x) = \frac{u_{43}(x)}{(\sqrt{x} + 1)^2},$$

with

$$u_{43}(x) = 3x^2 + 12x^{(3/2)} + 10x + 12\sqrt{x} + 3 + \sqrt{2x^2 + 2}(\sqrt{x} + 1)^2 - 3\sqrt{2x + 2}(\sqrt{x} + 1)^3.$$

Now we shall show that $u_{43}(x) \geq 0, \forall x > 0$. We shall apply twice the argument given in Part 1 of Theorem 2.1. Let us consider

$$\begin{aligned} v_{43}(x) &= \left(\frac{3x^2 + 12x^{(3/2)} + 10x + 12\sqrt{x} + 3}{+3 + \sqrt{2x^2 + 2}(\sqrt{x} + 1)^2} \right)^2 \\ &\quad - \left[3\sqrt{2x + 2}(\sqrt{x} + 1)^3 \right]^2 \\ &= 2\sqrt{2x^2 + 2} \left(\frac{3x^3 + 18x^{5/2} + 37x^2 + 44x^{3/2} + 37x + 18\sqrt{x} + 3}{+7x^4 + 28x^{7/2} + 72x^3 + 148x^{5/2} + 130x^2 + 148x^{3/2} + 72x + 28\sqrt{x} + 7} \right) \\ &\quad - \left(\frac{7x^4 + 28x^{7/2} + 72x^3 + 148x^{5/2} + 130x^2 + 148x^{3/2} + 72x + 28\sqrt{x} + 7}{+23x^7 + 518x^{13/2} + 3589x^6 + 13324x^{11/2} + 33239x^5 + 60922x^{9/2} + 85773x^4 + 96744x^{7/2} + 85773x^3 + 60922x^{5/2} + 33239x^2 + 13324x^{3/2} + 3589x + 518\sqrt{x} + 23} \right). \end{aligned}$$

Let us consider again

$$\begin{aligned} v_{43a}(x) &= \left[2\sqrt{2x^2 + 2} \left(\frac{3x^3 + 18x^{5/2} + 37x^2 + 44x^{3/2} + 37x + 18\sqrt{x} + 3}{+7x^4 + 28x^{7/2} + 72x^3 + 148x^{5/2} + 130x^2 + 148x^{3/2} + 72x + 28\sqrt{x} + 7} \right) \right]^2 \\ &\quad - \left(\frac{7x^4 + 28x^{7/2} + 72x^3 + 148x^{5/2} + 130x^2 + 148x^{3/2} + 72x + 28\sqrt{x} + 7}{+23x^7 + 518x^{13/2} + 3589x^6 + 13324x^{11/2} + 33239x^5 + 60922x^{9/2} + 85773x^4 + 96744x^{7/2} + 85773x^3 + 60922x^{5/2} + 33239x^2 + 13324x^{3/2} + 3589x + 518\sqrt{x} + 23} \right)^2 \\ &= (\sqrt{x} - 1)^2 \left(\frac{+23x^7 + 518x^{13/2} + 3589x^6 + 13324x^{11/2} + 33239x^5 + 60922x^{9/2} + 85773x^4 + 96744x^{7/2} + 85773x^3 + 60922x^{5/2} + 33239x^2 + 13324x^{3/2} + 3589x + 518\sqrt{x} + 23}{+23x^7 + 518x^{13/2} + 3589x^6 + 13324x^{11/2} + 33239x^5 + 60922x^{9/2} + 85773x^4 + 96744x^{7/2} + 85773x^3 + 60922x^{5/2} + 33239x^2 + 13324x^{3/2} + 3589x + 518\sqrt{x} + 23} \right). \end{aligned}$$

Thus the non-negativity of $v_{43a}(x)$ proves that $v_{43}(x) \geq 0, \forall x > 0$, thereby proving that $u_{43}(x) \geq 0, \forall x > 0$. This completes the proof of the result. □

Remark 3.1. As a consequence of above results we have the following new inequalities:

$$(i) \quad 4D_{N_3N_1} \leq I \leq \frac{2}{3} \left\{ \frac{D_{P_5P_4}}{D_{SP_4}} \right\};$$

$$(ii) \quad \frac{2}{5}D_{SP_4} \leq I \leq \frac{2}{3}D_{SP_4};$$

$$(iii) \quad \frac{2}{3}h \leq I \leq h;$$

$$(iv) \quad I \leq \frac{1}{8}J \leq T \leq \frac{1}{4}J.$$

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